Interplays between Harper and Mathieu equations

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This paper deals with the application of relationships between Harper and Mathieu equations to the derivation of energy formulas. Establishing suitable matching conditions, one proceeds by inserting a concrete solution to the Mathieu equation into the Harper equation. For this purpose, one resorts to the nonlinear oscillations characterizing the Mathieu equation. This leads to the derivation of two kinds of energy formulas working in terms of cubic and quadratic algebraic equations, respectively. Combining such results yields quadratic equations to the energy description of the Harper equation, incorporating all parameters needed.

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I. INTRODUCTION

Nonlinear oscillations described by the celebrated Mathieu equation [1]

$$\frac{d^2x}{dt^2}+\omega_0^2 x=f(x,\nu t)=\lambda x\cos(\nu t+\delta), \tag{1}$$

where ω_0 stands for eigenfrequency and where the other parameters and variables are self-consistently understood have received much interest. First, there are connections with several recent developments, such as the trapping of charged and neutral particles [2,3], or the continuum limit of the Harper equation [4]. Furthermore, the radial Schrödinger equation with an $1/r^4$ interaction can be converted into a Mathieu equation [5], and the same concerns the wave equation in elliptical coordinates [6]. Concerning Eq. (1), some recent studies, like resonances in the dynamics of kinks perturbed by ac forces [7] or interplays between nonlinearity and instability in nonautonomous oscillators [8] are worthy of being mentioned. There are reasons to say that Mathieu and Harper equations are still important in various areas of physics, but the interest on the Harper equation is even increasing nowadays. Indeed, we are able to mention some remarkable advances, such as statistics of resonances and delay times [9], the role of the fractal energy spectrum in the description of the generalized Hall conductance [10,11], or the duality between the Harper equation and the two-dimensional (2D) d-wave superconductivity with a magnetic field [12]. Moreover, proofs have been given [13] that the spectral determinant of the Harper equation generates the logarithm of the partition function of the 2D Ising model, as well as the asymptotic bandwidth formula [14,15]. Such spectacular issues do not mean, of course, that the Mathieu equation is less fundamental. Indeed, proofs have been given quite recently that the stability properties of D branes can be formulated successfully by invoking, once again, the Mathieu equation [16]. So, both equations exhibit unexpected capabilities of dealing with many problems, but there are several aspects that are still subject to a more adequate theoretical understanding. We shall then use this opportunity to perform a certain step along this direction, now by analyzing in some more detail mutual relationships between the Harper and Mathieu equations. Although Eq. (1) is well understood from a general mathematical point of view [1,6], we would like to discuss the concrete form of its solutions in terms of the asymptotic description of nonlinear oscillations [17]. The next step is to establish the conditions under which such concrete solutions proceeding to second order, i.e., $x(t)=x_2(t)$, are able to be implemented into the Harper equation

$$\varphi(n+1)+\varphi(n-1)+2\Delta\cos(n\hbar^*+\delta)\varphi(n)=E\varphi(n). \tag{2}$$

To this aim, a generalized version of Eq. (2) will be analyzed in some more detail. Here, n denotes an integer, $\varphi(n)$ is the wave function, Δ stands for a gap parameter, whereas $\beta=\hbar^*/2\pi$ has the meaning of a commensurability parameter [18]. We have also assumed, for convenience, that both interaction terms in Eqs. (1) and (2) are characterized by the same phase-parameter δ. The implementation of the $x_2(t)\sim\varphi(n)$ solution referred to above, proceeds of course, by using a suitable discretization of the time parameter. This produces different kinds of explicit energy formulas, which are useful, even if in a somewhat alternative manner, for a deeper understanding of large $\hbar^*$-scale behaviors and for further applications.

This paper is organized as follows. In Sec. II, one deals with preliminaries and basic ideas. The nonlinear oscillations relying on the Mathieu equation are derived in Sec. III. The first kind of energy solutions is derived by virtue of matching conditions in Sec. IV. In Sec. V, one deals with the implementation of nonlinear oscillations into the Harper equation. Proceeding in this manner yields further energy formulas, as shown in Sec. VI. Extra compatibility conditions concerning such energies are discussed in Sec. VII. The algebraic equations to the derivation of the energy are also able themselves to be combined together. Quadratic algebraic equations to the energy description of the Harper equation encompassing all parameters may then be easily written down. Conclusions are presented in Sec. VIII. Some basic formulas to the asymptotic description of nonlinear oscillations are presented in the Appendix.

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II. PRELIMINARIES AND BASIC IDEAS

Starting from Eq. (2) and assuming that $n \gg 1$ produces the continuous approximation

$$\frac{d^2\varphi(n)}{dn^2}+(2-E)\varphi(n)=-2\Delta\cos(n\hbar^*+\delta)\varphi(n), \quad (3)$$

which relies on Eq. (1) by virtue of a differential transformation rule between the corresponding independent variables such as

$$dn=f_s dt, \quad (4)$$

in which $f_s=f_s(E,\Delta,\delta)$ has the role of a rescaling parameter. This yields matching conditions such as $\omega_0^2=f_s^2(2-E)$ and $\lambda=-2f_s^2\Delta$. The conversion of $\nu t+\delta$ into $n\hbar^*+\delta$, and conversely remains to be done by invoking a relevant nondifferential discrete realization of the relationship between n and t. For this purpose, we have to recall that the leading contribution to the nonlinear oscillations characterizing Eq. (1) exhibits the typical form $x_t(t)=a\cos\psi$, where the amplitude is denoted by ''a'' and where the phase ψ is given by

$$\psi=\psi(t)=\omega_{\mathrm{eff}}t+\psi_0. \quad (5)$$

The pertinent effective frequency $\omega_{\mathrm{eff}}=\omega_{\mathrm{eff}}(\nu,\omega_0,\lambda)$ can be easily calculated using, e.g., available formulas proceeding to second ε order [17].

In this context, we have to realize that an almost natural time-discretization condition reads

$$\omega_{\mathrm{eff}}t+\psi_0=2n\pi+\alpha, \quad (6)$$

which serves as a transmutation condition, i.e., as a constraint to the realization of the mapping of Eq. (1) into Eq. (2). Accordingly, one has

$$f_s=\frac{dn}{dt}=\frac{\omega_{\mathrm{eff}}}{2\pi}. \quad (7)$$

Moreover, one finds

$$\nu t+\delta=2\pi n\frac{\nu}{\omega_{\mathrm{eff}}}+\frac{\nu}{\omega_{\mathrm{eff}}}(\alpha-\psi_0)+\delta, \quad (8)$$

which generates in turn the $n\hbar^*+\delta$-contribution one looks for as soon as

$$\nu=h_i^* f_s=\frac{\hbar^*}{2\pi}\omega_{\mathrm{eff}}, \quad (9)$$

and $\alpha=\psi_0$.

On the other hand, resorting to the Bloch factorization

$$\varphi(n)=e^{i\theta_1 n}\zeta(n), \quad (10)$$

yields the modified Harper equation (see also Ref. [19])

$$e^{i\theta_1}\zeta(n+1)+e^{-i\theta_1}\zeta(n-1)+2\Delta\cos(n\hbar^*+\delta)\zeta(n) = E\zeta(n), \quad (11)$$

which reproduces Eq. (2), as soon as $\theta_1=0$. We have to say that Eq. (11) serves specifically to the description of Bloch electrons on a two-dimensional lattice threaded by a transversal and homogeneous magnetic field. The corresponding wave vectors and lattice spacings are denoted by k_i and a_i $(i=1,2)$, respectively, whereas $\theta_1=k_1a_1$ but $\theta_2=\delta=k_2a_2$. Then, $\beta=\phi/\phi_0$ is the number of magnetic flux quanta per unit cell $(\phi_0=h/e)$. Resorting to the discretization done by Eq. (6) and considering Taylor-series expansions relying again on $n \gg 1$, one finds modified matching conditions such as

$$\omega_0^2=f_s^2\left(2-\frac{E}{\cos\theta_1}\right), \quad (12)$$

and

$$\lambda=-\frac{2f_s^2\Delta}{\cos\theta_1}, \quad (13)$$

provided that

$$\sin\theta_1\frac{d}{dn}\zeta(n)=0. \quad (14)$$

This corresponds to selected stationary $x_2(t)$ solutions if $\sin\theta_1\neq 0$, so that we then have to account for the equivalent $dx_2/dt=0$ condition. It is understood that dealing with Eq. (11), the former implementation $x_2(t)\to\varphi(n)$ has to be replaced by $x_2(t)\to\zeta(n)$.

Focusing our attention on Eqs. (1) and (11) we shall then proceed as follows. First, using the effective frequency ω_{eff} we shall derive an energy solution, i.e., $E=E_1(\hbar^*,\theta_1,\Delta)$, by applying solely Eqs. (9), (12), and (13). Next, we shall insert the concrete $x_2(t)$ solution to Eq. (1) into Eq. (11), but now in the context of arbitrary finite n values. This insertion works in terms of Eq. (6), but residual imaginary terms have to be ruled out, as this time, one works without invoking Taylor-series expansions corresponding to the $n \gg 1$ choice.

Proceeding in conjunction with Eq. (14) and assuming that $\sin\theta_1\neq 0$, yields a second energy-solution $E=E_2(\theta_1,\theta_2,\hbar^*)$, but under extra relationships such as $\Delta=\Delta(\hbar^*,\theta_1,\theta_2)$. This matter concerns Eqs. (35), (44), and (51). Putting together $E_1(\hbar^*,\theta_1,\Delta)$ and $E_2(\hbar^*,\theta_1,\theta_2)$ results in energy candidates incorporating all parameters bearing on Eq. (11), as shown by Eq. (55). We have also to remark that Eq. (11) produces the intermediary energy representation

$$E=2\cos\theta_1+2\Delta\cos(n\hbar^*+\delta), \quad (15)$$

if again $n \gg 1$, where the last term remains to be specified later [see Eq. (38)]. So far, Eq. (15) serves to the identification of the energy range as

$$E \in [2\cos\theta_1 - 2\Delta, 2\cos\theta_1 + 2\Delta] \subseteq [-2-2\Delta, 2+2\Delta], \tag{16}$$

so that $E \in [-4,4]$ if $\Delta = 1$. On the other hand, it is well known that the exact energy solutions to the Harper equation are real ones. Being aware that the present results are approximations, we shall then look for the real parts of energies mentioned above, by checking both role and magnitude of imaginary parts. It is also well known that the spectrum of Eq. (2) exhibits rich and interesting structures [20,21], as displayed by the famous Hofstadter butterfly (see Fig. 1 in Ref. [20]).

III. ASYMPTOTIC APPROACH TO NONLINEAR OSCILLATIONS

Applying general approximation formulas established before (see the Appendix) we have to say that nonlinear oscillations characterizing Eq. (1) within the nonresonance regime are given by

$$x(t) = x_2(t) = a\cos\psi + \varepsilon U_1(a,\psi,\nu t), \tag{17}$$

to second ε order, where

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a), \tag{18}$$

and

$$\frac{d\psi}{dt} = \omega_0 + \varepsilon B_1(a) + \varepsilon^2 B_2(a). \tag{19}$$

The small ε parameter serves to the introduction of pertinent power-series expansions, which also means that we have to put $\varepsilon = 1$ at the end of calculations. Resorting to the double Fourier-series expansions

$$f_0(a,\psi,\nu t) = f(a\cos\psi,\nu t) = \sum_{n,m} f^{(0)}_{n,m}(a)\exp i(n\nu t + m\psi), \tag{20}$$

one readily finds that the only nonzero coefficients are

$$f^{(0)}_{1,1} = f^{(0)}_{1,-1} = f^{(0)*}_{-1,1} = f^{(0)*}_{-1,-1} = \frac{\lambda a}{4}\exp(i\delta), \tag{21}$$

where the star denotes complex conjugation. These solutions are subject to the nonresonance condition $\omega_0^2 \neq (n\nu + m\omega_0)^2$. Accordingly,

$$U_1(a,\psi,\nu t) = \frac{\lambda a}{\nu(4\omega_0^2 - \nu^2)}[\nu\cos\psi\cos(\nu t + \delta) + 2\omega_0\sin\psi\sin(\nu t + \delta)]. \tag{22}$$

In addition, $A_1(a) = A_2(a) = B_1(a) = 0$, but

$$\frac{d\psi}{dt} = \omega_0 - \frac{\varepsilon^2\lambda^2}{16\pi^2\omega_0(4\omega_0^2 - \nu^2)}. \tag{23}$$

This enables us to introduce the effective frequency as

$$\frac{d\psi}{dt} = \omega_{\rm eff}(\nu,\omega_0,\lambda) = \omega_0\left[1 + \frac{\lambda^2\varepsilon^2}{8\pi^2\omega_0^2(4\omega_0^2 - \nu^2)}\right]^{-1/2}, \tag{24}$$

so that

$$\psi = \psi_2(t) = \omega_{\rm eff}(\nu,\omega_0,\lambda)t + \psi(0). \tag{25}$$

Under such conditions, the nonlinear oscillations characterizing Eq. (1) are given by

$$x(t) = x_2(t) = a_0\cos\psi_2(t) + U_1(a_0,\psi_2(t),\nu t), \tag{26}$$

where $a(t) = a(0) = a_0$, which proceeds both to second ε order and within the nonresonance regime mentioned above.

Next, it can be verified that $x_2(t)$ is stationary if

$$\frac{\lambda}{4\omega_0^2 - \nu^2}\left[2\omega_0 - \omega_{\rm eff} + \frac{2\omega_0}{\nu^2}(2\omega_0\omega_{\rm eff} - \nu^2)\right] = \omega_{\rm eff}\left(1 + \frac{4\omega_0^2}{\nu^2}\tan^2\alpha\right)^{1/2}. \tag{27}$$

One realizes immediately that Eq. (27) has the root

$$\tan\alpha = \frac{1}{2\nu\omega_0}(\lambda^2 - \nu^4)^{1/2}, \tag{28}$$

which can be rewritten equivalently as

$$\tan\alpha = \frac{\nu}{2\omega_0}\left(\frac{4\Delta^2}{\hbar^{*4}\cos^2\theta_1} - 1\right)^{1/2}, \tag{29}$$

in accord with Eqs. (9) and (13).

IV. THE FIRST TYPE OF ENERGY RESULTS

Using Eqs. (9), (12), and (13) yields the cubic algebraic equation

$$2\cos^2\theta_1\left(2 - \frac{E}{\cos\theta_1}\right)\left(2 - \frac{E}{\cos\theta_1} - \frac{\hbar^{*2}}{4}\right)\times\left(2 - \frac{E}{\cos\theta_1} - 4\pi^2\right) - \Delta^2 = 0, \tag{30}$$

for which explicit energy solutions such as $E = E_1(\hbar^*,\theta_1,\Delta)$ can be easily established. Such cubic equations exhibit, of course, either three real roots or two complex-conjugated roots supplemented by a real one. It is also clear that such concrete root realizations are sensitive with respect to the values of underlying parameters. Using hereafter the notations $x = \hbar^*$, $y = E$, $z = \Delta$, and $u_1 = \cos\theta_1$ and assuming, e.g., that $z = 1$ and $u_1 = 1$, yields the roots

$$y_1(x) = \frac{1}{12}s_1^{1/3} - 12 s_2 s_1^{-1/3} - \frac{4}{3}\pi^2 - \frac{x^2}{12} + 2, \tag{31}$$

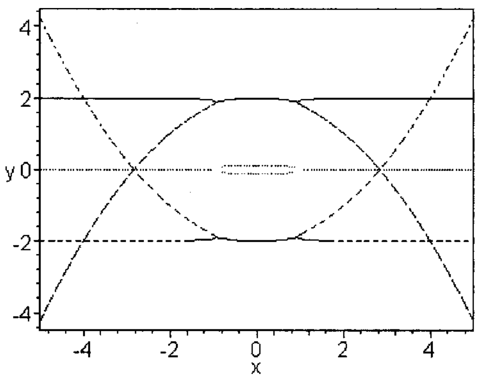

FIG. 1. The x dependence of $y=y_1$ (solid curve) and $y=y_3$ (dashed curve) for $u_1=1$ and $z=1$. The closed-dotted curve on the x axis indicates the I interval within which the above roots are complex. So, y_i ($i=1,3$) stands for $\mathrm{Re}\, y_i$ if $x\in I$, too. The dashed curve crosses the x axis at $x\cong \pm 2.833$. The $u_1=-1$ counterparts of these energies are displayed by dot-dashed curves.

and

$$y_3(x)=-\frac{1}{12}s_1^{1/3}\varepsilon_+ +12 s_2 s_1^{-1/3}\varepsilon_- -\frac{4}{3}\pi^2-\frac{x^2}{12}+2, \tag{32}$$

where $\varepsilon_\pm=(1\pm i\sqrt{3})/2$. One has

$$s_1=-432+384\pi^4x^2+24\pi^2x^4-4096\pi^6-x^6+12(384\pi^6x^6 -3072\pi^8x^4-12\pi^4x^8+1296-144\pi^2x^4-2304\pi^4x^2 +6x^6+25576\pi^6)^{1/2}, \tag{33}$$

and

$$s_2=\frac{1}{9}x^2\pi^2-\frac{16}{9}\pi^4-\frac{x^4}{144}. \tag{34}$$

There is also a third root that is real, i.e. $y_2(x)$, which comes from $y_3(x)$ by replacing i by $-i$. This latter root varies quite slowly with x as $y_2(x)\cong -37.48$ for $x\in[-5,5]$. So, it is located far away from the typical energy-domain $y\in[-4,4]$ characterizing the Harper equation. The x dependence of $y_1(x)$ and $y_3(x)$ is displayed in Fig. 1. The x dependence of the imaginary parts of these roots is represented by the closed patched curve located on the x axis. So, the imaginary parts are nonzero only for $x\in I$, where $I\cong[-0.95,0.95]$. Accordingly, there is $\mathrm{Re}\, y_1=\mathrm{Re}\, y_3$ for $x\in I$, but $y_1>y_3$ if $x\notin I$. In particular, $y_1(0)\cong 2.000\,16+0.112\,54i$, which shows that the amplitude of imaginary parts is rather small. Moreover, the width of the I interval decreases with z, and the same concerns the amplitude of imaginary parts. Of course, the above energies change their sign if one inserts $u_1=-1$ instead of $u_1=1$. It should be remarked that $u_1=\pm 1$ energies discussed above would be invoked exclusively in so far as $dx_2/dt\neq 0$, but we shall see later that reasonable solutions to $dx_2/dt=0$ may also be proposed.

V. IMPLEMENTING NONLINEAR OSCILLATIONS INTO THE HARPER EQUATION

Inserting Eq. (26) into Eq. (11) and accounting for the mutual interconnection between $n\pm 1$ and $t\pm 2\pi/\omega_{\mathrm{eff}}$ yields the algebraic equation

$$x^2u_1^2\left(2-\frac{y}{u_1}-\frac{x^2}{4}\right)\left[x^2+s\left(2-\frac{y}{u_1}\right)\right] -z^2\left(2\cos x-\frac{y}{u_1}+sx^2\right)=0, \tag{35}$$

by virtue of the identification $\zeta(n)\sim x_2(t)$, provided that

$$\tan(nx+\delta)\equiv\tan(\nu t+\delta)=\frac{2\omega_0}{\nu}\tan\alpha>0. \tag{36}$$

This latter condition is responsible for the cancelling of imaginary contributions that are implied by the substitution just mentioned above. For this purpose, matching conditions done by Eqs. (9), (12), and (13) have been used in conjunction with Eq. (6).

Using Eqs. (29) and (36) leads to the relationship

$$\tan(nx+\delta)=\Omega(x,z,u_1)=\left(\frac{4z^2}{x^4u_1^2}-1\right)^{1/2}, \tag{37}$$

which plays the role of a consistency condition, where $u_2=\cos\theta_2=\cos\delta$. Correspondingly, there is

$$\cos(nx+\delta)=s\,\frac{u_1x^2}{2z}, \tag{38}$$

which is responsible for the $s=\pm 1$ parameter mentioned above. In order to handle Eq. (37), we shall start from the assumption

$$x=\frac{\pi}{2}n_r, \tag{39}$$

where $|n_r|=1,2,3,\ldots,$ One would then obtain $Q=4P/n_r$ for rational values of the commensurability parameter such as $\beta=P/Q$. It is obvious that Eq. (39) works safely at least for $|n_r|=1$, 2, and 4, i.e., for $Q=\pm 4P$, $Q=\pm 2P$, and $Q=\pm P$. It is also clear that we have to consider that $|n_r|\leqslant 4$ if $|x|\leqslant 2\pi$, as assumed usually. Inserting Eq. (39) into Eq. (37) gives

$$\tan\theta_2=\Omega(x,z,u_1), \tag{40}$$

and

$$-\cot\theta_2=\Omega(x,z,u_1), \tag{41}$$

for even and odd values of the nn_r product, respectively. We may remark that Eqs. (40)–(41) could be viewed as some generalized versions of the well-known matching conditions for the symmetric square-well potential. Accordingly,

$$z=z_1(n_r,u_1,u_2)=\frac{\pi^2}{8}n_r^2\left|\frac{u_1}{u_2}\right|, \tag{42}$$

and

$$z=z_2(n_r,u_1,u_2)=\frac{\pi^2}{8}n_r^2\frac{|u_1|}{\sqrt{1-u_2^2}}, \tag{43}$$

where $u_2=\cos\theta_2=\cos\delta$. For convenience, we have restricted ourselves to positive z values. Under such circumstances, Eqs. (42)–(43) have to be understood as a quantized version of the z solution to Eq. (37). Conversely, we have to realize that the continuous counterpart of the z solution just presented above may be obtained by invoking once again Eq. (39). This yields

$$z=z_3(x,u_1,u_2)=\frac{x^2}{2}\left|\frac{u_1}{u_2}\right|, \tag{44}$$

and

$$z=z_4(x,u_1,u_2)=\frac{x^2}{2}\frac{|u_1|}{\sqrt{1-u_2^2}}, \tag{45}$$

respectively. Of course, Eqs. (43) and (45) are valid if $|u_2|\neq 1$. It should be remarked that starting with $s=\mathrm{sgn}(u_1/u_2)$, Eqs. (38) and (44) produce rational values of the commensurability parameter such as

$$\beta=\frac{x}{2\pi}=\frac{n'}{n}, \tag{46}$$

and even more specifically, as

$$\beta=\frac{x}{2\pi}=\frac{2n'+1}{2n}, \tag{47}$$

if $z=z_3$, such that $\mathrm{sgn}(u_2)=1$ and $\mathrm{sgn}(u_2)=-1$, respectively. Here, n and n' are arbitrary integers. In particular, Eq. (47) is able to be used in the description of typical $\beta=P/Q$ values for which P is odd and Q is even. Accordingly, we can say that Eq. (44) has to be favored, but for the sake of generality, Eqs. (42), (43), and (45) deserve also to be studied under appropriate conditions. We also have to remark that Eqs. (42)–(45) are invariant under the substitutions $z\rightarrow 1/z$, $u_1\rightarrow u_1/z^2$, and $u_2\rightarrow u_2$, which have the meaning of certain scaling-duality transformations Then, Eqs. (30) and (35) would be invariant themselves under such transformations if $y\rightarrow y/z^2$. This differs, however, from the well-known Aubry duality, for which $y\rightarrow y/z$ [22,23]. Then, $z=1$ is a self-dual point that corresponds to a metal-insulator transition [22].

Under such circumstances, Eq. (37) becomes itself tractable, but the tradeoff is a mutual interconnection between the parameters z, x, u_1, and u_2. In general, this may proceed in the presence of the quantum number n_r introduced above or not. Such interconnections and/or parameter fixings are not at all an absolute novelty. Indeed, similar relationships have been used in the case of conditionally exactly solvable potentials [24], for several exactly solvable systems [25] or for eigenvalue problems, which can be solved with the help of a supersymmetric description [26,27]. The general understanding is that under such interconnections, the systems one deals with become solvable, which amounts to the onset of dynamical symmetries [28–30].

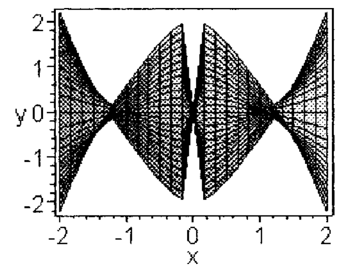

FIG. 2. The u_1 section view of the three-dimensional plot characterizing the $y=y_{-1}^-$ solution for $z=1$, $x\in[-2,2]$, and $u_1\in[-1,1]$.

VI. CONCRETE DERIVATIONS OF ENERGY FORMULAS

Looking for typical behaviors, let us present the energy solutions, i.e., $y=y_s^{\pm}(u_1,z,x)$, implied solely by Eq. (35). Inserting, e.g., $z=1$, one obtains the solutions

$$y_s^{\pm}=\frac{1}{8u_1x^2}(f_1\pm\sqrt{g_1}), \tag{48}$$

where

$$f_1=(4s-1)u_1^2x^4+16u_1^2x^2-4s, \tag{49}$$

and

$$g_1=(17+8s)u_1^4x^8+(32+8s)u_1^2x^4 -128su_1^2x^2(1-\cos x)+16. \tag{50}$$

An appealing illustration of the capabilities of Eq. (48) is presented in Fig. 2. We can say that the u_1-section view displayed by y_{-1}^- exhibits nearly apparent fingerprints of the Hofstadter butterfly. However, this time one has positive- and negative-energy gaps involved symmetrically by small or near-to-zero values of the x parameter.

Coming back to Eq. (37), one gets faced with substitutions like $z=z_j$, as indicated by Eqs. (42)–(45). Ruling out the z parameter from Eq. (35) results in four quadratic algebraic equations to the evaluation of the energy

$$4u_2^2\left(2-\frac{y}{u_1}-\frac{x^2}{4}\right)\left[x^2+s\left(2-\frac{y}{u_1}\right)\right]-\frac{4u_2^2z_j^2}{u_1^2x^2}\times\left(2\cos x-\frac{y}{u_1}+sx^2\right)=0, \tag{51}$$

where $j=1$, 2, 3, and 4. The $y=y_{j,s}^{\pm}(x,u_1,u_2)$ solutions can then be written down quite automatically. Choosing $j=3$ and $u_2=1$, one finds the energies

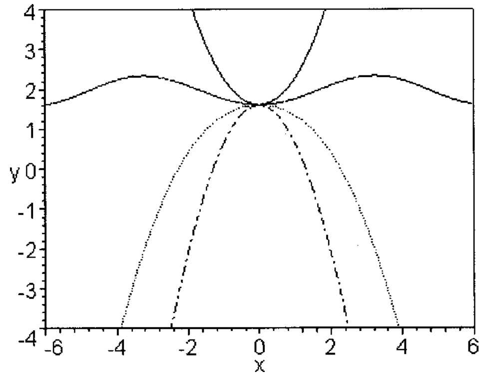

FIG. 3. The x dependence of $y^{+}_{3,1}$ (solid curve), $y^{-}_{3,-1}$ (waved-dashed curve), $y^{-}_{3,1}$ (dotted curve), and $y^{+}_{3,-1}$ (dashed-dotted curve) for $u_2=1$ and $u_1=0.8$. All curves meet together at $y=1.6$ as $x=0$. The curves $y=y^{-}_{3,1}(x)$ and $y=y^{+}_{3,-1}(x)$ cross the x axis at $x\cong\pm 2.278$ and ± 1.30, respectively.

$$y^{\pm}_{3,s}=\frac{1}{8s}\{x^2(3-s)+16s \pm u_1[x^4(26+10s+32sx^2[\cos(x)-1]^{1/2}]\}. \quad (52)$$

The x dependence of corresponding energies is displayed in Fig. 3 for $u_1=0.8$ and $u_2=1$. The waved curve in Fig. 3, i.e., the x dependence of $y^{-}_{3,-1}$, may be viewed as being reminiscent to oscillating structures characterizing actual Harper energies, but this time one deals with a distorted symmetry center that is moved upwards at $x=0$ and $y=1.6$. Note that the remaining solutions done by Eq. (52) are located again beyond the $y\in[-4,4]$ domain of interest. The u_1-section view produced by the three-dimensional plot of $y^{-}_{3,1}(u_1,x)$ is presented in Fig. 4. Comparing Figs. 2 and 4, one sees that the energy gaps involved in the former case have been removed by virtue of the regular behavior of present solutions at $x=0$. The other cases can be treated in a similar manner.

VII. COMBINATIONS AND FURTHER PERSPECTIVES

Next, we have to say, strictly speaking, that second type of energy solutions presented above have to fulfil Eq. (30). This results in certain compatibility relationships concerning underlying parameters. Such relationships are numerically involved ones, so that we shall consider here just a first-degree realization of Eq. (51), namely, the one corresponding to $j=3$ and $u_2=0$. One would then obtain the energy

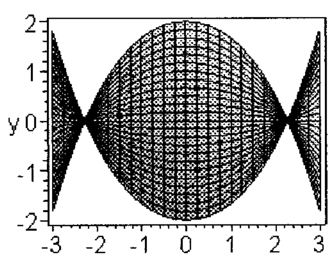

FIG. 4. The u_1 section view of the three-dimensional plot concerning the x and u_1 dependence of $y=y^{-}_{3,1}$ for $u_2=1$.

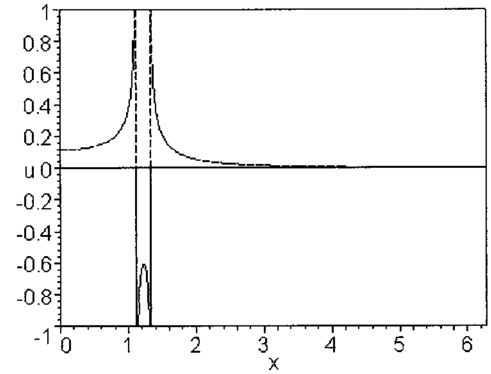

FIG. 5. The x dependence of $u=\mathrm{Re}\,u_{1,+}$ (solid curve) and $u=\mathrm{Im}\,u_{I,+}$ (dotted curve) for $x\in[0,2\pi]$.

$$y=y_s=u_1(2\cos x+sx^2), \quad (53)$$

which can be viewed as an updated version in Eqs. (15) and (38). Inserting Eq. (53) into Eq. (30), we then have to establish the conditions under which reasonable u_1-roots become realized. One finds the roots

$$u_{1,\pm}=\pm z[2F_1(F_1-4\pi^2)(F_1-x^2/4)]^{-1/2}, \quad (54)$$

where $F_1=2-y_s/u_1$. The x dependence of $\mathrm{Re}\,u_{1,+}$ and $\mathrm{Im}\,u_{1,+}$ is presented in Fig. 5 for $z=s=1$. One sees that there is a clean nonzero u_1 solution within the "window" $x\in I_{\mathrm{SC}}\cong[1.142,1.306]$ only, where the subscript SC stands for "strong compatibility." Correspondingly, $-1\lesssim u_{1,+}\lesssim -0.608$, whereas $\mathrm{Im}\,u_{1,+}=0$ if $x\in I_{\mathrm{SC}}$. However, $\mathrm{Im}\,u_{1,+}\neq 0$ if $x\notin I_{\mathrm{SC}}$, but $\mathrm{Im}\,u_{1,+}\lesssim 0.01$ if $x\gtrsim 3.4$. This means that $\mathrm{Im}\,u_{1,+}$ may be neglected for sufficiently large x values, in which case $u_{1,+}\cong 0$ represents a reasonable approximation. On the other hand, one has $\mathrm{Im}\,u_1=0$ if $x\gtrsim 2\pi$, but now for $s=-1$. This yields again a safe u_1 solution, but underlying x values are excessively large. So far, we then have to say that similar investigations could also be done in other cases, but a noticeable fulfillment of SC relationships remains questionable.

Now, let us remember that matching conditions underlying Eq. (30) have also been used in the derivation of the second type of energy formula discussed above. Accordingly, extra compatibility requirements expressed in terms of the simultaneous fulfillment of Eqs. (30) and (51) can be interpreted as being too hard. Looking for reasonably tractable formulations, we shall then resort to certain algebraic combinations between Eqs. (30) and (51). The easiest way to deal with this is to rewrite equivalently these equations, such that, e.g., both factors $(2-y/u_1)$ and $(2-y/u_1-x^2/4)$ get located within the same side. Performing the quotient of corresponding expressions then gives the four quadratic equations

$$\left(2-\frac{y}{u_1}-4\pi^2\right)\left[z_j^2\left(2\cos x-\frac{y}{u_1}+sx^2\right)-u_1^2x^4\left(2-\frac{y}{u_1}-\frac{x^2}{4}\right)\right]-\frac{2}{s}x^2z^2=0, \quad (55)$$

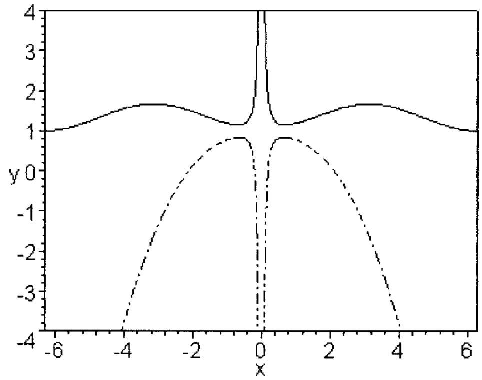

FIG. 6. The x dependence of $y=y_{3,-1,+}$ (solid-waved curve) and $y=\mathrm{Re}\,y_{3,1,+}$ (dot-dashed curve) for $u_1=0.5$ and $z=u_2=1$. Both curves diverge as $x\to 0$.

where $j=1$, 2, 3, and 4, which can be again easily solved. This time the roots look like $y=y_{j,s,\pm}(u_1,u_2,z,x)$, so that all parameters are accounted for. Choosing once again $j=3$ and $u_2=1$, one obtains the energies

$$y_{3,s,\pm}=f_3\pm\frac{1}{6}\sqrt{g_3}, \qquad (56)$$

where

$$f_3=-\frac{s+1}{6}u_1x^2-\frac{u_1}{3}\cos x+\frac{7}{3}u_1-2u_1\pi^2, \qquad (57)$$

and

$$g_3=4u_1^2(\cos x-1)[x^2(1+s)-12\pi^2]-\frac{24s}{x^2}z^2+144\pi^4u_1^2 + 4u_1^2(\cos x-1)^2+2u_1^2x^2(1+s)(x^2-12\pi^2). \qquad (58)$$

The undesirable feature is that the z^2/x^2 term produces a singularity at $x=0$. The x dependence of $y_{3,-1,+}$ and of $\mathrm{Re}\,y_{3,1,+}$ is displayed in Fig. 6 for $u_1=0.5$ and $u_2=z=1$. Note that $y_{3,1,+}$ becomes complex for $|x|<0.083$. The interesting point is that $s=-1$ solutions $y_{3,-1}^{-}$ and $y_{3,-1,+}$ are, up to the singularity at $x=0$, quite similar, as shown by the waved curves in Figs. 3 and 6. Conversely, this similarity is able to favor Eq. (51).

After having arrived at this stage, we are in a position to propose an approximate but ''nonlinear'' alternative to the secular-equation method for the study of the Harper equation [31,32]. Indeed, using Eq. (55), we can establish an expression for $\cos x$ in terms of a rational function, i.e., $R_{21}(y)$, as follows

$$\cos x\equiv R_{21}(y)=\frac{y}{2u_1}-s\frac{x^2}{2}+2u_2^2\left(2-\frac{y}{u_1}-\frac{x^2}{4}\right) + \frac{su_2^2z^2}{u_1^2x^2}\left(2-\frac{y}{u_1}-4\pi^2\right)^{-1}. \qquad (59)$$

Next, resorting to rational values of the commensurability parameter such as $\beta=P/Q$, where now P and Q are mutually prime integers, one has

$$\cos(Qx)=P_Q(\cos x)=P_Q(R_{21}(y))=1, \qquad (60)$$

where $P_Q(\cos x)$ denotes the usual Q-degree polynomial representation of $\cos(Qx)$ with respect to $\cos x$. What then remains is to solve Eq. (60) for fixed values of the Q parameter. So, one finds, e.g., solutions proceeding via selected $\cos x$ inputs such as $\pm\sqrt{2}/2$, $(0,\pm\sqrt{3}/2)$, and $\pm(1/2\pm\sqrt{2}/4)$ for $Q=2$, 3, and 4, respectively. In general, the plots produced by Eq. (60) are similar to the ones presented in Fig. 6, but other details remain open for further investigations. Equation (35), as it stands, may also be treated in a similar manner.

VIII. CONCLUSIONS

In this paper, sensible manifestations of interplays between Mathieu and Harper equations have been established and discussed. Resorting to the nonlinear oscillations characterizing the Mathieu equation, we found a cubic equation for the energy, such as that in Eq. (30) and quadratic equations, as shown in Eqs. (35) and (51). In order to derive Eq. (35), the $x_2(t)$ solution to the Mathieu equation has been implemented into the Harper equation by virtue of the transmutation condition in Eq. (6), but matching conditions expressed by Eqs. (9), (12), and (13) have also been invoked. This differs from the cubic equation mentioned above, for which only matching conditions have been used. Energy candidates incorporating all parameters may also be proposed. To this aim, a first-version synthesis of the two kinds of energy formulas has also been written down, as shown in Eq. (55). Extra compatibility relationships between Eqs. (30) and (51) could be able to provide interesting information, but for this purpose, complicated numerical studies have to be done. Accordingly, second kind type of energy solutions based on Eqs. (35) and (36) should also be invoked irrespective of Eq. (30). Combining Eqs. (59) and (60) leads again to quadratic equations to the derivation of energy formulas incorporating, besides Q, all parameters needed. It is also clear that the present wave-function $\zeta(n)\sim x_2(t)$ exhibits the periodic boundary-condition $\zeta(n+Q)=\zeta(n)$ in accord with Eq. (26), provided that $\beta=P/Q$. A Q-band free energy may also be easily established, thus establishing a useful starting point for further applications. Such results would then be able to provide a better understanding of present energy results, too.

Also worthy of being referred to is the so-called energy-reflection (ER) symmetry [33]. This means that besides E, there is also a solution with the opposite sign. This symmetry has been discussed in an explicit manner with respect to a selected middle-band description, such as that used in the derivation of a symmetrized version of the Harper-equation [34]. The ER symmetry is not valid automatically in more general cases, even if certain conversions of the original Harper equation into the symmetrized one have been proposed [35]. Concrete realizations of such symmetrized energy solutions have also been established [36]. Concerning the present results, we have to realize that there are selected parameter domains where the symmetry referred to above is exhibited in an effective manner, as shown by the plots presented in Figs. 1, 2, and 4. It should also be noted that all algebraic energy equations discussed in this paper, i.e., Eqs. (30), (35), (51), and (55), are invariant under scaling-duality transformations presented in Sec. V.

We have to recognize that quickly tractable energies discussed above have, rather, the meaning of leading approximations, so that they are able to reflect certain large-scale manifestations of the Hofstadter butterfly [20]. However, this does not prevent us from establishing nontrivial energy profiles, which are able to provide a better understanding of several details and interconnections. In this context, we have to realize that bifurcations that are seen in the Harper spectrum are able to be understood in terms of present parameter-dependent transitions from complex to real roots (see Fig. 1). Moreover, such bifurcations, closed loops included, serve as basic elements to the formation of complex patterns. Further improvements could be done by resorting to a more refined description of the nonlinear oscillations characterizing the Mathieu equation. We also have to remark that all algebraic equations can be solved in terms of the shifted energy variable $F=2-E/u_1$ instead of $y=E$. Squaring Eq. (12), we then have to come back to the usual energy via $E=(2\mp F)u_1$, which yields, of course, a doubling of solutions.

It is understood that the energy candidates produced by Eqs. (30), (35), (51), (53), and (55) are not at all highly accurate generic solutions to the Harper equation, such as studied before by several authors (see, e.g., Refs. [37–40]). They represent, in fact, selected energy profiles along which the Harper wave functions are transmutation by-products of nonlinear oscillations described by Eq. (26). Nevertheless, the interesting point is that such energy curves may be established in a well-defined manner, both for $u_1^2=1$ and $u_1^2\neq 1$. Of course, in the first case both Eqs. (30) and (35) work as they stand. It should also be noted that Eq. (1) exhibits a $\nu=\omega_0$ resonance to second ε order, which can be treated, under some more specific assumptions, in a similar way.

Summarizing, we may then say that our main emphasis in this paper was on the investigation of mutual relationships between the Mathieu and Harper equations. Though less accurate, the present energy-profile studies are able to serve for a better understanding of large-scale behaviors characterizing the Harper equation, now by using explicit energy formulas. We believe that such alternative studies have their own interest from a general theoretical point of view. In this context, we also learned how to combine Eq. (4), i.e., a differential-geometric relationship, with a discrete transmutation condition such as seen in Eq. (6).

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APPENDIX

The asymptotic description of nonlinear oscillations is based on Eqs. (17)–(19), such as produced, e.g., by Eqs. (13.30)–(13.37) in Ref. [17]. Replacing $f(x,\nu t)$ by $\varepsilon f(x,\nu t)$ and inserting Eqs. (17)–(19) into Eq. (1), we then have to proceed order by order. There is $x=a\cos\psi$ to first ε order, such that

$$A_1(a)=-\frac{1}{4\pi^2\omega_0}\int_0^{2\pi}d\theta'\int_0^{2\pi}d\psi f(a\cos\psi,\theta')\sin\psi, \tag{A1}$$

and

$$B_1(a)=-\frac{1}{4\pi^2\omega_0 a}\int_0^{2\pi}d\theta'\int_0^{2\pi}d\psi f(a\cos\psi,\theta')\cos\psi, \tag{A2}$$

where θ' stands for νt. These equations reflect the so-called harmonic balance. Next, x is given by Eq. (17) to second ε order, such that

$$A_2(a)=-\frac{1}{2\omega_0}\left[aA_1\frac{dB_1}{da}+2A_1B_1\right]-\frac{1}{4\pi^2\omega_0}\int_0^{2\pi}d\theta'\int_0^{2\pi}d\psi f_1(a,\psi,\theta')\sin\psi, \tag{A3}$$

and

$$B_2(a)=\frac{1}{2a\omega_0}\left[A_1\frac{dA_1}{da}+aB_1^2\right]-\frac{1}{4\pi^2\omega_0 a}\int_0^{2\pi}d\theta'\int_0^{2\pi}d\psi f_1(a,\psi,\theta')\cos\psi. \tag{A4}$$

One has

$$f_1(a,\psi,\nu t)=U_1(a,\psi,\nu t)\frac{\partial}{\partial x}f(x,\nu t)\bigg|_{x=a\cos\psi}, \tag{A5}$$

where

$$U_1(a,\psi,\nu t)=\sum_{n,m}\frac{f_{n,m}^{(0)}(a)}{\omega_0^2-(n\nu+m\omega_0)^2}\exp i(n\nu t+m\psi), \tag{A6}$$

works in accord with Eq. (20). The fact that now the right-hand side of Eq. (1) is independent of dx/dt has also been accounted for. This approach has been referred to in the literature as the Krylov-Bogoliubov-Mitropolsky method (see, e.g., Chap. 14 in Ref. [41]). An updated version of this method has also been formulated by incorporating algebraic symmetries [42].

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